

AD-A103 866

WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER  
THE INITIAL BOUNDARY VALUE PROBLEM FOR THE EQUATIONS OF MOTION --ETC(U)  
JUL 81 A MATSUMURA, T NISHIDA  
DAAG29-80-C-0041

F/G 20/4

UNCLASSIFIED

MRC-TSR-2237

NL

| 4 |  
A  
81-8401

END  
DATE  
FILED  
O 81  
DTIC

AD A103866

MRC Technical Summary Report # 2237

THE INITIAL BOUNDARY VALUE PROBLEM  
FOR THE EQUATIONS OF MOTION OF  
COMPRESSIBLE VISCOUS AND  
HEAT-CONDUCTIVE FLUID

Akitaka Matsumura and Takaaki Nishida

Mathematics Research Center  
University of Wisconsin-Madison  
610 Walnut Street  
Madison, Wisconsin 53706

July 1981

(Received May 1, 1981)

DTIC  
ELECTED  
SEP 8 1981  
S D  
A

Approved for public release  
Distribution unlimited

Sponsored by

U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

National Science Foundation  
Washington, DC 20550

81 9 08 038

UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

THE INITIAL BOUNDARY VALUE PROBLEM FOR THE EQUATIONS  
OF MOTION OF COMPRESSIBLE VISCOUS AND  
HEAT-CONDUCTIVE FLUID

Akitaka Matsumura\* and Takaaki Nishida\*\*

Technical Summary Report #2237

July 1981

ABSTRACT

We prove the global existence and uniqueness of solutions to the equations of motion for compressible, viscous and heat-conductive Newtonian fluid in a bounded domain, with small initial data and external force, and boundary conditions of zero velocity and constant temperature. We also show that the solution decays exponentially to a unique equilibrium state. The proof uses an energy method similar to the one used in our previous results on the pure initial value problem plus some new techniques for estimates near the boundary.

AMS (MOS) Subject Classifications: 35B40, 35K60, 35M05, 76N10.

Key Words: Initial boundary value problem, Equations of motion, Compressible viscous and heat-conductive fluid, Global solutions in time, Energy method.

Work Unit Number 1 - Applied Analysis

---

\* Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606, Japan.

\*\* Department of Mathematics, Kyoto University, Kyoto 606, Japan.

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062.

#### SIGNIFICANCE AND EXPLANATION

The motion of compressible, viscous and heat-conductive Newtonian fluid is described by a system of partial differential equations which is of hyperbolic-parabolic type and highly nonlinear. Having previously treated the pure initial value problem for these equations (see, e.g., MRC TSR #1991, 2194), we now consider the initial boundary value problem for motion within a smooth bounded container whose walls are kept at constant temperature, subject to a conservative external force, such as gravity. For this problem we prove the existence of a unique smooth solution for all positive time, and we show that this solution must decay exponentially to a unique equilibrium state. Since the system is quasilinear with respect to the unknowns, density, velocity and temperature, we need to assume that the initial data are close to the equilibrium state, and that the external force is sufficiently small.

Accession For	
NTIS	GRA&I <input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/ _____	
Availability Codes	
Dist	Avail and/or Special
A	

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE INITIAL BOUNDARY VALUE PROBLEM FOR THE EQUATIONS  
OF MOTION OF COMPRESSIBLE VISCOUS AND  
HEAT-CONDUCTIVE FLUID

Akitaka Matsumura\* and Takaaki Nishida\*\*

**§1. Introduction. Statement of Theorem.**

The motion of a compressible, viscous and heat-conductive Newtonian fluid is described by five conservation laws:

$$(1.1) \quad \left\{ \begin{array}{l} \rho_t + (\rho u^j)_{x_j} = 0, \\ u^i_t + u^j u^i_{x_j} + \frac{1}{\rho} p_{x_i} = \frac{1}{\rho} (\mu (u^i_{x_j} + u^j_{x_i}) + \mu' (u^k_{x_k}) \delta^{ij})_{x_j} \\ \quad + f^i, \quad i = 1, 2, 3, \\ \theta_t + u^j \theta_{x_j} + \frac{\theta p \theta}{\rho c_v} u^j_{x_j} = \frac{1}{\rho c_v} ((\kappa \theta_{x_j})_{x_j} + \Psi), \end{array} \right.$$

where  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ ,  $t > 0$ ,  $\rho$  is the density,  $u = (u^1, u^2, u^3)$  is the velocity,  $\theta$  is the absolute temperature,  $p = p(\rho, \theta)$  is the pressure,  $f = (f^1, f^2, f^3)$  is the external force,  $\mu = \mu(\rho, \theta)$  and  $\mu' = \mu'(\rho, \theta)$  are viscosity coefficients,  $\kappa = \kappa(\rho, \theta)$  is the coefficient of heat conductivity,  $c_v = c_v(\rho, \theta)$  is the heat capacity at constant volume and  $\Psi = \frac{1}{2} \frac{\mu}{\rho} (u^j_{x_k} + u^k_{x_j})^2 + \mu' (u^j_{x_j})^2$  is the dissipation function. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^3$  with a smooth boundary  $\partial\Omega$ . We consider the initial boundary value problem for (1.1) in  $\Omega$  with the boundary conditions

\* Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606, Japan.

\*\*

Department of Mathematics, Kyoto University, Kyoto 606, Japan.

$$(1.2) \quad u(t, x) = 0, \quad \theta(t, x) = \bar{\theta}, \quad x \in \partial\Omega, \quad t > 0 ,$$

where  $\bar{\theta}$  is any fixed positive constant; and with initial data

$$(1.3) \quad (\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x), \quad x \in \Omega .$$

We seek a classical solution for (1.1) ~ (1.3) globally in time.

Historically, a local solution in time for (1.1) ~ (1.3) was constructed by Tani [6]. On the other hand, Kazhikov and Shelukhin [3] proved the existence and uniqueness of a global solution in time for the one-space-dimensional model of (1.1) ~ (1.3). In this paper, we prove the existence and uniqueness of a global solution in time for the original problem (1.1) ~ (1.3) if the initial data and external force are suitably small. Although the proof is given by an energy method similar to those of our previous papers [4] and [5] on the initial value problem, the initial boundary value problem requires now a priori estimates of the solution near the boundary  $\partial\Omega$ . In order to state the main theorem precisely, let us list our assumptions:

- (i) The external force  $f^i$  is generated by a potential function  $\Phi(x)$ , i.e.,

$$f^i = -\Phi_{x_i} .$$

- (ii)  $\mu, \mu', \kappa, p, c_v \in C^\infty(0)$ , where

$$\Omega = \{(\rho, \theta) \mid (\rho, \theta) \in (0, +\infty) \times (0, +\infty)\}.$$

- (iii)  $\mu, \kappa, p, p_\rho, p_\theta, c_v > 0$  and  $\mu' + \frac{2}{3}\mu > 0$  for  $(\rho, \theta) \in \Omega$ .

- (iv)  $(\rho_0, u_0, \theta_0) \in H^4(\Omega)$ ,  $\Phi \in H^5(\Omega)$  and  $(\rho_0, \theta_0)(x) > 0$  for  $x \in \bar{\Omega}$ .

- (v) (compatibility condition)

$$u_0|_{\partial\Omega} = 0, \quad \theta_0|_{\partial\Omega} = \bar{\theta},$$

$$-(p_0)_{x_i} + (\mu_0(u_0)_{x_j}^i + u_0)_{x_i}^j + u_0 u_0_{x_k}^k \delta^{ij})_{x_j} - p_0 \Phi_{x_j}|_{\partial\Omega} = 0,$$

$$-\theta_0(p_\theta)_{x_j}^j + (\kappa_0 \theta_0)_{x_j}^j + \Psi_0|_{\partial\Omega} = 0,$$

where  $p_0 = p(p_0, \theta_0)$ ,  $u_0 = u(p_0, \theta_0), \dots$ , and so on. Here  $H^k(\Omega)$  denotes the Sobolev's space on  $\Omega$  with the norm  $\|\cdot\|_k$ , and  $H_0^k(\Omega)$  denotes the completion of  $C_0^\infty(\Omega)$  in  $H^k(\Omega)$ . Define  $\|\cdot\|_{C^k}$  by

$$\|u\|_{C^k} = \max_{|\alpha| \leq k} \sup_{x \in \Omega} \left| \left( \frac{\partial}{\partial x} \right)^\alpha u(x) \right| ,$$

and also define  $\bar{\rho}$  by

$$\bar{\rho} = \frac{1}{V(\Omega)} \int_{\Omega} \rho_0(x) dx ,$$

where  $V(\Omega)$  represents the volume of  $\Omega$ . We say that  $(\tilde{\rho}(x), \tilde{u}(x), \tilde{\theta}(x)) \in C^1(\bar{\Omega}) \times (C^2(\bar{\Omega}))^2$  is an equilibrium state of the problem (1.1) ~ (1.3) when  $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$  satisfies (1.1) and the following additional conditions

$$(1.4) \quad \tilde{u}|_{\partial\Omega} = 0 , \quad \tilde{\theta}|_{\partial\Omega} = \bar{\theta} , \quad \frac{1}{V(\Omega)} \int_{\Omega} \tilde{\rho} dx = \bar{\rho} .$$

Then we have the following main theorem.

**Theorem 1.1.** Under the assumptions (i) ~ (v), there exist positive constants  $\epsilon$ ,  $\alpha$  and  $C$  such that if  $\|\rho_0 - \bar{\rho}\|_0, \|u_0 - \bar{u}\|_4 + \|\theta_0 - \bar{\theta}\|_5 < \epsilon$ , the problem (1.1) ~ (1.3) has a unique global solution in time  $(\rho, u, \theta)$  and a unique equilibrium state  $(\tilde{\rho}, \tilde{u}, \tilde{\theta}) = (\tilde{\rho}, 0, \bar{\theta})$  satisfying

$$\begin{aligned} \rho &\in C^0(0, +\infty; H^4(\Omega)) \cap C^1(0, +\infty; H^3(\Omega)) , \\ (u, \theta) &\in C^0(0, +\infty; H^4(\Omega)) \cap C^1(0, +\infty; H^2(\Omega)) , \end{aligned}$$

and

$$\|(\rho, u, \theta)(t) - (\tilde{\rho}, 0, \bar{\theta})\|_{C^0} \leq Ce^{-\alpha t} .$$

## §2. Existence and Uniqueness of the Equilibrium State.

Let us rewrite the equations and conditions for the equilibrium state  $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ :

$$(2.1) \quad (\tilde{\rho} \tilde{u}^j)_{x_j} = 0 ,$$

$$(2.2) \quad \tilde{\rho} u_{x_j}^j + \tilde{p}_{x_i} + \tilde{\mu} \Phi_{x_i} - (\tilde{\mu} u_{x_j}^i + \tilde{u}_{x_i}^j) + \tilde{\mu} u_{x_k}^k \delta^{ij} \Big|_{x_j} = 0 ,$$

$$(2.3) \quad \tilde{\rho} c_v u_{x_j}^j \tilde{\theta}_{x_j} + \tilde{\theta} p_{\theta} u_{x_j}^j - (\tilde{\kappa} \tilde{\theta}_{x_j})_{x_j} - \tilde{\psi} = 0 ,$$

$$(2.4) \quad \tilde{u}|_{\partial\Omega} = 0 , \quad \tilde{\theta}|_{\partial\Omega} = \bar{\theta} , \quad \int \tilde{\rho} dx = \int \bar{\rho} dx ,$$

where  $\tilde{p} = p(\tilde{\rho}, \tilde{\theta})$ ,  $\tilde{\mu} = \mu(\tilde{\rho}, \tilde{\theta}), \dots$ , and so on. Then we have

**Lemma 2.1.** Given the assumptions (i) ~ (v), there exist positive constants  $\epsilon$  and  $C$  such that if  $\|\Phi\|_1 < \epsilon$ , the problem (2.1) ~ (2.4) has a unique solution  $(\tilde{\rho}(x), 0, \bar{\theta})$  in a small neighborhood of  $(\bar{\rho}, 0, \bar{\theta})$  in  $C^1(\bar{\Omega}) \times (C^2(\bar{\Omega}))^2$  satisfying

$$\|\tilde{\rho} - \bar{\rho}\|_1 < C \|\Phi\|_1 ,$$

where  $\tilde{\rho}(x)$  is determined by

$$(2.5) \quad \begin{cases} \text{Const.} - \Phi(x) = \frac{\tilde{\rho}(x)}{\bar{\rho}} \frac{p_{\rho}(s, \bar{\theta})}{s} ds , \\ \int \tilde{\rho}(x) - \bar{\rho} dx = 0 . \end{cases}$$

Furthermore we have  $\tilde{\rho} \in H^5(\Omega)$  and

$$(2.6) \quad \|\tilde{\rho} - \bar{\rho}\|_5 < C \|\Phi\|_5 \text{ for } \|\Phi\|_5 < \epsilon .$$

**Proof of Lemma 2.1.** Suppose  $|\rho - \bar{\rho}|, |u|, |\theta - \bar{\theta}| < \frac{1}{2} \min(\bar{\rho}, \bar{\theta})$ . Then we may estimate the equalities

$$(2.7) \quad \begin{cases} \int [2.1] \times \int \frac{\tilde{\rho} p_{\rho}(s, \bar{\theta})}{\bar{\rho}} ds dx = 0 , \\ \int (\tilde{\rho} u^i) [2.2]^i dx = 0 , \\ \int \tilde{\rho} (\tilde{\theta} - \bar{\theta}) [2.3] dx = 0 , \end{cases}$$

where [2.1], [2.2]<sup>i</sup> and [2.3] denote the left hand side of (2.1), (2.2)<sup>i</sup> and (2.3) respectively. After integrating (2.7) by parts and using Poincare's inequalities

$$(2.8) \quad \|u\| \leq C\|Du\|, \quad \|\theta - \bar{\theta}\| \leq C\|D\theta\|,$$

one can show that

$$(2.9) \quad \|Du\|^2, \|D\theta\|^2 \leq C(\|D\rho\|_{C^0} + \|\theta - \bar{\theta}\|_{C^0})(\|Du\|^2 + \|D\theta\|^2).$$

Here and in what follows, we denote by  $D_z^k f$  for  $k > 0$  and  $z = (z_1, \dots, z_m)$  the vector function

$$D_z^k f = \{\partial_z^k f / \partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \dots \partial z_m^{\alpha_m} \text{ for all } |\alpha| = k\}$$

and in particular write  $D^k f$  when  $z = x$ . Therefore, if  $\|D\rho\|_{C^0}$  and

$\|\theta - \bar{\theta}\|_{C^0}$  are suitably small, it follows from (2.4) and (2.9) that

$$(2.10) \quad \tilde{u} \equiv 0, \quad \tilde{\theta} \equiv \bar{\theta}.$$

Substituting (2.10) into (2.2), we have

$$(2.11) \quad \left( \int_0^\rho \frac{p_p(s, \bar{\theta})}{s} ds + \Phi \right)_{x_i} = 0,$$

which implies (2.5) immediately.

One may use the implicit function theorem once to see that the constant in (2.5) depends on  $\Phi$  as a smooth functional on  $H^5(\Omega)$ , and again to prove (2.6).

### §3. Local and Global Existence.

First let us rewrite the problem (1.1) ~ (1.3) by the change of variables  $(\rho, u, \theta) \rightarrow (\rho + \tilde{\rho}, u, \theta + \bar{\theta})$  as follows:

$$(3.1) \quad \left\{ \begin{array}{l} \rho_t + u^j \rho_{x_j} + (\rho + \tilde{\rho}) u^j_{x_j} + \rho_{x_j} u^j = 0 , \\ u_t^i + u^j u_{x_j}^i - \frac{1}{\rho + \tilde{\rho}} (\mu (u_{x_j}^i + u_{x_i}^j) + \mu' \mu_{x_k}^k \delta^{ij})_{x_j} + \\ + \frac{p_\rho}{\rho + \tilde{\rho}} \rho_{x_i} + \frac{p_\theta}{\rho + \tilde{\rho}} \theta_{x_i} = \frac{\tilde{p}_\rho (\rho + \tilde{\rho}, \theta + \bar{\theta})}{(\rho + \tilde{\rho}) p_\rho (\rho, \theta)} - 1 ) \theta_{x_i} , \\ \theta_t + u^j \theta_{x_j} + \frac{(\theta + \bar{\theta}) p_\theta}{(\rho + \tilde{\rho}) c_v} u^j_{x_j} = - \frac{1}{(\rho + \tilde{\rho}) c_v} ((\kappa \theta_{x_j})_{x_j} + \Psi) , \end{array} \right.$$

$$(3.2) \quad (u, \theta)|_{\partial\Omega} = 0 ,$$

$$(3.3) \quad (\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x) .$$

Furthermore we rewrite the problem (3.1) ~ (3.3) as follows:

$$(3.4)^0 \quad L_u^0(\rho, u) \equiv \rho_t + u^j \rho_{x_j} + \bar{\rho} u^j_{x_j} = f^0 ,$$

$$(3.4)^i \quad L^i(\rho, u, \theta) \equiv u_t^i - \mu u_{x_j x_j}^i - (\mu + \mu') u_{x_i x_j}^j + \\ + p_1 \rho_{x_i} + p_2 \theta_{x_i} = f^i ,$$

$$(3.4)^4 \quad L^4(u, \theta) \equiv \theta_t - \kappa \theta_{x_j x_j} + p_3 u_{x_j}^j = f^4 ,$$

$$(3.5) \quad (u, \theta)|_{\partial\Omega} = 0, \quad (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0) ,$$

where define  $L_v^0(\rho, u) \equiv \rho_t + v^j \rho_{x_j} + \bar{\rho} u^j_{x_j}$  and  $\mu = \bar{\mu}/\bar{\rho}$ ,  $\mu' = \bar{\mu}'/\bar{\rho}$ ,

$p_1 = \bar{p}_\rho/\bar{\rho}$ ,  $p_2 = \bar{p}_\theta/\bar{\rho}$ ,  $p_3 = \bar{\theta} \bar{p}_\rho/\bar{\rho} \bar{c}_v$ ,  $\kappa = \bar{\kappa}/\bar{\rho} \bar{c}_v$ ,  $\bar{\mu} = \mu(\bar{\rho}, \bar{\theta})$ ,  $\bar{\mu}' = \mu'(\bar{\rho}, \bar{\theta})$ ,  $\bar{p}_\rho = p_\rho(\bar{\rho}, \bar{\theta})$  and so on, and

$$f^0(\rho, u, \theta) \equiv (\bar{\rho} - \rho - \tilde{\rho}) u_{x_j}^j - \tilde{\rho}_{x_j} u^j ,$$

$$\begin{aligned}
f^1(\rho, u, \theta) &\equiv u^j u^i_{x_j} + \left( \frac{\mu}{\rho + \tilde{\rho}} - \frac{\bar{\mu}}{\tilde{\rho}} \right) u^i_{x_j x_j} + \\
&+ \left( \frac{\mu + \mu'}{\rho + \tilde{\rho}} - \frac{\bar{\mu} + \bar{\mu}'}{\tilde{\rho}} \right) u^j_{x_i x_j} + \frac{1}{\rho + \tilde{\rho}} (u^i_{x_j} (u^i_{x_j} + u^j_{x_i}) + \\
&+ (u^i_{x_j} u^k_{x_k} \delta^{ij}) + \left( \frac{\bar{\rho}}{\tilde{\rho}} - \frac{\bar{\rho}_p}{\rho + \tilde{\rho}} \right) \rho_{x_i} + \\
&+ \left( \frac{\bar{\rho}_\theta}{\tilde{\rho}} - \frac{\rho_\theta}{\rho + \tilde{\rho}} \right) \theta_{x_i} + \left( \frac{\bar{\rho} p_p (\rho + \tilde{\rho}, \theta + \bar{\theta})}{(\rho + \tilde{\rho}) p_p (\rho, \theta)} - 1 \right) \phi_{x_i} , \\
f^4(\rho, u, \theta) &\equiv -u^j \theta_{x_j} + \left( \frac{\kappa}{(\rho + \tilde{\rho}) c_v} - \frac{\bar{\kappa}}{\tilde{\rho} c_v} \right) \theta_{x_j x_j} + \\
&+ \left( \frac{\bar{\theta} \bar{\rho}_\theta}{\tilde{\rho} c_v} - \frac{(\theta + \bar{\theta}) p_\theta}{(\rho + \tilde{\rho}) c_v} \right) u^j_{x_j} + \frac{1}{(\rho + \tilde{\rho}) c_v} (\kappa_{x_j} \theta_{x_j} + \Psi) .
\end{aligned}$$

Next, by Sobolev's lemma there exists a positive constant  $E_0$  such that

$$\|f\|_{C^0} < \frac{1}{2} \min(\bar{\rho}, \bar{\theta}) \text{ for } \|f\|_2 < E_0.$$

Then the solution of (3.1) ~ (3.3) is sought in the set of functions

$x(0, +\infty; E)$  for some  $E < E_0$ , where for  $0 < t_1 < t_2 < +\infty$ ,

$$x(t_1, t_2; E) = \{(\rho, u, \theta)\}$$

$$\rho \in C^0(t_1, t_2; H^4) \cap L^2(t_1, t_2; H^4) ,$$

$$\rho_t \in C^0(t_1, t_2; H^3) \cap L^2(t_1, t_2; H^3) ,$$

$$\rho_{tt} \in C^0(t_1, t_2; H^1) \cap L^2(t_1, t_2; H^2) ,$$

$$(u, \theta) \in C^0(t_1, t_2; H^4 \cap H_0^1) \cap L^2(t_1, t_2; H^5) ,$$

$$(u, \theta)_t \in C^0(t_1, t_2; H^2 \cap H_0^1) \cap L^2(t_1, t_2; H^3) ,$$

$$(u, \theta)_{tt} \in C^0(t_1, t_2; L^2) \cap L^2(t_1, t_2; H_0^1),$$

$$N(t_1, t_2) < E\}.$$

Here  $N(t_1, t_2)$  is defined by

$$\begin{aligned} N^2(t_1, t_2) = & \sup_{t_1 \leq t \leq t_2} (\|\rho(t)\|_4^2 + \|\rho_t(t)\|_3^2 + \|\rho_{tt}(t)\|_1^2 + \\ & + \|u(t)\|_4^2 + \|u_t(t)\|_2^2 + \|u_{tt}(t)\|_1^2) + \\ & + \int_{t_1}^{t_2} (\|\rho(\tau)\|_4^2 + \|\rho_t(\tau)\|_3^2 + \|\rho_{tt}(\tau)\|_2^2 + \|u(\tau)\|_5^2 + \\ & + \|u_t(\tau)\|_3^2 + \|u_{tt}(\tau)\|_1^2) d\tau. \end{aligned}$$

We can get a global solution by a combination of a local existence result and some a priori estimates for the solution in  $X$ .

**Proposition 3.1. (local existence)** Suppose the problem (3.1) ~ (3.3) has a unique solution  $(\rho, u, \theta) \in X(0, h; E_0)$  for some  $h > 0$  and then consider the problem (3.1) ~ (3.3) for  $t > h$ . Then there exist positive constants  $\tau, e_0$  and  $C_0$  ( $e_0 \sqrt{1+C_0^2} < E_0$ ) which are independent of  $h$  such that if  $N(h, h), \|\theta\|_5 < e_0$ , the problem has a unique solution

$$(\rho, u, \theta) \in X(h, h+\tau; C_0 N(h, h)).$$

**Proposition 3.2. (a priori estimate)** Suppose the problems (3.1) ~ (3.3) has a solution  $(\rho, u, \theta) \in X(0, h; E_0)$  for some  $h > 0$ . Then there exist positive constants  $e_1$  and  $C_1$  ( $e_1 < e_0, e_1 C_1 < E_0$ ) which are independent of  $h$  such that if  $N(0, h), \|\theta\|_5 < e_1$ , it holds that  $N(0, h) < C_1 N(0, 0)$ .

Given Propositions 3.1 and 3.2, the global existence uniqueness of solutions may be proved as follows. Choose the initial date  $(\rho_0, u_0, \theta_0)$  and the potential function  $\Phi$  so small that

$$N(0,0) \leq \min\left(e_0, \frac{e_1}{c_0}, \frac{e_1}{c_1 \sqrt{1+c_0^2}}\right),$$

and  $\|\phi\|_5 < e_1$ . Then Proposition 3.1 with  $h = 0$  gives a local solution

$$(\rho, u, \theta) \in X(0, \tau; C_0 N(0,0)),$$

Since  $C_0 N(0,0) \leq e_1 \leq e_0$ , Proposition 3.2 with  $h = \tau$  implies

$$N(0, \tau) \leq c_1 N(0,0),$$

and Proposition 3.1 with  $h = \tau$  implies

$$(\rho, u, \theta) \in X(\tau, 2\tau; C_0 N(\tau, \tau)),$$

$$\in X(0, 2\tau; \sqrt{1+c_0^2} N(0, \tau)).$$

Hence, since  $\sqrt{1+c_0^2} N(0, \tau) \leq c_1 \sqrt{1+c_0^2} N(0,0) \leq e_1$ , Proposition 3.2 with

$h = 2\tau$  implies

$$N(0, 2\tau) \leq c_1 N(0,0),$$

and Proposition 3.1 with  $h = 2\tau$  gives

$$(\rho, u, \theta) \in X(2\tau, 3\tau; C_0 N(2\tau, 2\tau)),$$

$$\in X(0, 3\tau; \sqrt{1+c_0^2} N(0, 2\tau)).$$

Repetition of this process yields:

Proposition 3.3 (global existence)

There exist positive constants  $e$  and  $C$  such that if  $N(0,0)$ ,

$\|\phi\|_5 < e$ , the problem (3.1) ~ (3.3) has a unique solution

$$(\rho, u, \theta) \in X(0, +\infty; C N(0,0)).$$

#### §4. Proof of Local Existence and Uniqueness.

We note only how to prove local existence and uniqueness for  $h = 0$ , and omit the details. Suppose  $(n, v, \zeta) \in X(0, \tau; E_0)$ . Then we first need to solve the following equations:

$$(4.1)^0 \quad L_v^0(\rho, u) = f^0(n, v, \zeta) ,$$

$$(4.1)^1 \quad L^1(\rho, u, \theta) = f^1(n, v, \zeta) ,$$

$$(4.4)^4 \quad L^4(u, \theta) = f^4(n, v, \zeta) ,$$

$$(4.2) \quad (u, \theta)|_{\partial\Omega} = 0, \quad (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0) ,$$

where we note that  $\{f_i^1\}_{i=0}^4$  satisfy the conditions

$$f^0 \in C^0(0, \tau; H^3) \cap L^2(0, \tau; H^4) ,$$

$$f_t^0 \in C^0(0, \tau; L^2) \cap L^2(0, \tau; H^2) ,$$

$$f_{tt}^0 \in L^2(0, \tau; L^2) ,$$

$$f^1 \in C^0(0, \tau; H^2) \cap L^2(0, \tau; H^3) ,$$

$$f_t^1 \in C^0(0, \tau; L^2) \cap L^2(0, \tau; H^1) ,$$

$$f_{tt}^1 \in C^0(0, \tau; H^{-1}) , \quad 1 \leq i \leq 4 ,$$

where  $H^{-1}$  represents the dual space of  $H_0^1$ .

To solve (4.1), (4.2), we may regard the equations (4.1) as

$$(4.2)^0 \quad \rho_t + v^j \rho_{x_j} = -\bar{\rho} u_{x_j}^j + f^0 ,$$

$$(4.2)^1 \quad u_t^i - \hat{\mu} u_{x_j x_j}^i - (\hat{\mu} + \hat{\mu}^*) u_{x_i x_j}^j = f^i - p_2 \theta_{x_1} - p_1 \rho_{x_1},$$

$$(4.2)^4 \quad \theta_t - \hat{\kappa} \theta_{x_j x_j} = f^4 - p_3 u_{x_j}^j,$$

and use standard arguments about first-order hyperbolic equations for  $(4.2)^0$   
and second-order parabolic systems for  $\{(4.2)^i\}_{i=1}^4$ . Once we get the solution  
of the linear problem (4.1)(4.2), we may construct an approximation sequence  
 $\{(\rho, u, \theta)^{(n)}\}_{n=0}^{+\infty}$  for the nonlinear problem (3.4)(3.5) as follows:

$$(\rho, u, \theta)^{(0)} \equiv (\rho_0, u_0, \theta_0),$$

and for  $n > 1$ ,

$$\left\{ \begin{array}{l} L^0_{u^{(n-1)}}(\rho^{(n)}, u^{(n)}) = f^0(\rho^{(n-1)}, u^{(n-1)}, \theta^{(n-1)}), \\ L^i(\rho^{(n)}, u^{(n)}, \theta^{(n)}) = f^i(\rho^{(n-1)}, u^{(n-1)}, \theta^{(n-1)}), \\ L^4(u^{(n)}, \theta^{(n)}) = f^4(\rho^{(n-1)}, u^{(n-1)}, \theta^{(n-1)}), \\ (u^{(n)}, \theta^{(n)})|_{\partial\Omega} = 0, \quad (\rho^{(n)}, u^{(n)}, \theta^{(n)})(0) = (\rho_0, u_0, \theta_0). \end{array} \right.$$

Finally, we may apply the contraction mapping principle to the nonlinear  
problem (3.4), (3.5) in the space  $X$  under the smallness conditions on the  
initial data and external force, to obtain existence and uniqueness of the  
solution.

§5. Proof of the A Priori Estimates.

Throughout this section, we suppose that  $\|\theta\|_5 \leq E_0$  and that the problem (3.1) ~ (3.3) has a solution  $(\rho, u, \theta) \in X(0, h; E_0)$  and we note that all constants are independent of  $h$ . First, recall Poincare's inequalities for  $(u, \theta)$ :

$$(5.1) \quad \|u\|^2 \leq C \|Du\|^2, \quad \|\theta\|^2 \leq C \|D\theta\|^2,$$

and the following lemma for  $\rho$ .

Lemma 5.1. It holds that

$$\|\rho\|^2 \leq C \|D\rho\|^2.$$

This is proved using the fact that the equation  $(3.1)^0$  implies

$$\int \rho(t) dx = \int \rho_0 dx = 0$$

for all  $t > 0$  (cf. [7]).

Next, since the viscosity and heat conduction terms of (3.4) define strongly elliptic operators on  $(u, \theta)$ , we have for  $(u, \theta) \in H^k \cap H_0^1$  ( $1 \leq k \leq 5$ ) (cf. [1], [2]):

$$(5.2) \quad \begin{aligned} \|u\|_k^2 &\leq C \sum_{i=1}^3 \|\hat{u}_{x_i x_j}\|_{k-2}^2 + (\hat{u} + \hat{u}^\ast)_{x_i x_j} \|_{k-2}^2, \\ \|\theta\|_k^2 &\leq C \|\hat{\theta}_{x_i x_j}\|_{k-2}^2. \end{aligned}$$

Equations (3.4) and estimates (5.1) and (5.2) easily yield the following lemma.

Lemma 5.2. For  $1 \leq k \leq 5$ , it holds that

$$\left\{ \begin{array}{l} \|u\|_k^2 \leq C(\|u_t\|_{k-2}^2 + \|\theta\|_{k-1}^2 + \|\rho\|_{k-1}^2 + \sum_{i=1}^3 \|f^i\|_{k-2}^2), \\ \|\theta\|_k^2 \leq C(\|\theta_t\|_{k-2}^2 + \|u\|_{k-1}^2 + \|f^4\|_{k-2}^2). \end{array} \right.$$

Third, the following estimates on the time derivatives of the solution over all of  $\Omega$  can be obtained rather easily.

Lemma 5.3. For any positive number  $\epsilon$  and  $0 < k < 2$ , it holds that

$$\begin{aligned} \|D_t^k(\rho, u, \theta)(t)\|^2 + \int_0^t \|D_t^k(u, \theta)\|_1^2 + \|D_t^k(\frac{d\rho}{dt})\|^2 d\tau \\ \leq C \|D_t^k(\rho, u, \theta)(0)\|^2 + \int_0^t \epsilon \|D_t^k \rho\|^2 + C(1 + \epsilon^{-1})(\|D_t^k f^0\|^2 + \\ + \sum_{i=1}^4 \|D_t^k f^i\|_{-1}^2) d\tau + CN^3(0, t), \end{aligned}$$

where  $\frac{d\rho}{dt}$  is defined by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u^j \frac{\partial \rho}{\partial x_j}.$$

This lemma is proved by estimating the following equation derived from (3.4):

$$\begin{aligned} & \int_0^t \int \frac{p_1}{\rho} D_t^k L^0 \cdot D_t^k \rho + D_t^k L^1 \cdot D_t^k u^i + \frac{p_2}{p_3} D_t^k L^4 \cdot D_t^k \theta dx d\tau \\ & = \int_0^t \frac{p_1}{\rho} D_t^k f^0 \cdot D_t^k \rho + D_t^k f^1 \cdot D_t^k u^i + \frac{p_2}{p_3} D_t^k f^4 \cdot D_t^k \theta dx d\tau, \end{aligned}$$

and noting from (3.4)<sup>0</sup> that

$$u^j_{x_j} = -\frac{1}{\rho} \frac{d\rho}{dt} + \frac{1}{\rho} f^0.$$

Fourth, let us establish the interior estimates. Let  $x_0$  be any fixed function in  $C_0^\infty(\Omega)$ . Then we have

Lemma 5.4. For  $0 < \ell < 3$ , it holds that

$$\begin{aligned} \|X_0 D_x^{\ell+1} \rho(t)\|^2 + \int_0^t \|X_0 D_x^{\ell+1} \rho\|^2 d\tau & \leq \\ & \leq C(\|\rho_0\|_{\ell+1}^2 + \|u_0\|_\ell^2 + \|u(t)\|_\ell^2 + \int_0^t \|u, \theta\|_{\ell+1}^2 + \end{aligned}$$

$$+ \| \varphi^0 \|_{k+1}^2 + \sum_{i=1}^3 \| \varphi^i \|_k^2 d\tau + N^3(0, t) .$$

**Lemma 5.5.** For any positive number  $\epsilon$  and  $1 \leq k \leq 4$ , it holds that

$$\begin{aligned} & \| x_0 D^k(\rho, u, \theta)(t) \|^2 + \int_0^t \| x_0 D^{k+1} u \|^2 + \| x_0 D^k (\frac{d\rho}{dt}) \|^2 d\tau \\ & \leq C \| \rho_0, u_0, \theta_0 \|_k^2 + \int_0^t \epsilon \| x_0 D^k \rho \|^2 + C(1 + \epsilon^{-1}) \\ & \quad \| u, \theta \|_k^2 + \| \varphi^0 \|_k^2 + \sum_{i=1}^4 \| \varphi^i \|_{k-1}^2 d\tau + CN^3(0, t) . \end{aligned}$$

To prove these, we may estimate for Lemma 5.4 the equation

$$\begin{aligned} & \int_0^t \int x_0^2 (D^k L^0)_{x_i} \cdot D^k \rho_{x_i} + \frac{\bar{\rho}}{2\hat{\mu} + \hat{\mu}'} x_0^2 D^k L^1 \cdot D^k \rho_{x_i} dx d\tau \\ & - \int_0^t \int x_0^2 D^k f^0_{x_i} \cdot D^k \rho_{x_i} + \frac{\bar{\rho}}{2\hat{\mu} + \hat{\mu}'} x_0^2 D^k f^1 \cdot D^k \rho_{x_i} dx d\tau , \end{aligned}$$

and for Lemma 5.5, the equation

$$\begin{aligned} & \int_0^t \int \frac{p_1}{\bar{\rho}} x_0^2 D^k L^0 \cdot D^k \rho + x_0^2 D^k L^1 \cdot D^k u^1 + \frac{p_2 x_0^2}{p_3} D^k L^4 \cdot D^k \theta dx d\tau \\ & - \int_0^t \int \frac{p_1}{\bar{\rho}} x_0^2 D^k f^0 \cdot D^k \rho + x_0^2 D^k f^1 \cdot D^k u^1 + \frac{p_2 x_0^2}{p_3} D^k f^4 \cdot D^k \theta dx d\tau . \end{aligned}$$

Fifth, let us establish the estimates near the boundary. We choose a finite number of bounded open sets  $\{O_i\}_{i=1}^N$  in  $R^3$  such that

$$\bigcup_{j=1}^N O_j = \partial\Omega ,$$

and in each open set  $O_j$  we choose local coordinates  $(\psi, \varphi, r)$  as follows:

(i) the surface  $O_j \cap \partial\Omega$  is the image of smooth functions

$y^i = y^i(\psi, \varphi)$  satisfying

$$(5.3) \quad \begin{cases} y_\psi^i \cdot y_\psi^i = 1, & y_\psi^i \cdot y_\varphi^i = 0, \\ y_\varphi^i \cdot y_\varphi^i > \delta > 0, \end{cases}$$

where  $\delta$  is some positive constant independent of  $1 \leq j \leq N$ ,

(ii) any  $x^i$  in  $O_j$  are represented by

$$(5.4) \quad x^i = x^i(\psi, \varphi, r) = r n^i(\psi, \varphi) + y^i(\psi, \varphi),$$

where  $n^i(\psi, \varphi)$  represents the internal unit normal vector at the point of the surface coordinated  $(\psi, \varphi)$ .

Here and in what follows, we omit the suffix  $j$  for simplicity. Let us define the unit vectors  $e_1^i$  and  $e_2^i$  by

$$e_1^i = y_\psi^i, \quad e_2^i = y_\varphi^i / |y^i|.$$

Then note that there exist smooth functions of  $(\psi, \varphi)$ ,  $(\alpha, \beta, \gamma)(\psi, \varphi)$  and  $(\alpha^i, \beta^i, \gamma^i)(\psi, \varphi)$  satisfying

$$\frac{\partial}{\partial \psi} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix} = \begin{pmatrix} 0 & -\gamma & -\alpha \\ \gamma & 0 & -\beta \\ \alpha & \beta & 0 \end{pmatrix} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix},$$

$$\frac{\partial}{\partial \varphi} \begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \end{pmatrix} = \begin{pmatrix} 0 & -\gamma^i & -\alpha^i \\ \gamma^i & 0 & -\beta^i \\ \alpha^i & \beta^i & 0 \end{pmatrix} \begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \end{pmatrix}.$$

An elementary calculation will show that the Jacobian  $J$  of the transformation (5.4) is

$$(5.6) \quad J = |\mathbf{x}_\phi \times \mathbf{x}_\varphi| \\ = |\mathbf{y}_\varphi| + (\alpha|\mathbf{y}_\varphi| + \beta^*)r + (\alpha\beta^* - \beta\alpha^*)r^2 .$$

By (5.6), we can see the transformation (5.4) is regular choosing  $r$  small if needed. Therefore the functions  $(\psi, \varphi, r)_{x_i}(x)$  make sense and are calculated as

$$(5.7) \quad \left\{ \begin{array}{l} \psi_{x_1} = \frac{1}{J} (x_\varphi \times x_r)_1 = \frac{1}{J} (Ae_1^i + Be_2^i) \\ \varphi_{x_1} = \frac{1}{J} (x_r \times x_\varphi)_1 = \frac{1}{J} (Ce_1^i + De_2^i) \\ r_{x_1} = \frac{1}{J} (x_\psi \times x_\varphi)_1 = n_1 \end{array} \right. ,$$

where  $A = |y_0| + \beta^*r$ ,  $B = -ra^*$ ,  $C = -\beta r$ ,  $D = 1 + ar$  and  $J = AD - BC > 0$ .

Hence (5.7) implies

$$\frac{\partial}{\partial x_1} = \frac{1}{J} (Ae_1^1 + Be_2^1) \frac{\partial}{\partial \psi} + \frac{1}{J} (Ce_1^1 + De_2^1) \frac{\partial}{\partial \varphi} + n^1 \frac{\partial}{\partial r} .$$

Thus, in each  $O_j$ , we can rewrite the equations  $\{(4.1)^i\}_{i=0}^3$  in the local coordinates  $(\psi, \varphi, r)$  as follows:

$$\rho_t + \frac{1}{J} ((Ae_1^i n^i + Be_2^i u^i) \rho_\psi + (Ce_1^i + De_2^i) u^i \rho_\varphi + u^i n^i J \rho_r) \\ + \frac{\bar{\rho}}{J} ((Ae_1^i + Be_2^i) u_\psi^i + (Ce_1^i + De_2^i) u_\varphi^i + J n^i u_r^i) = f^0$$

$$u_t^i + \frac{\hat{\mu}}{J^2} ((A^2 + B^2) u_\psi^i + 2(AC + BD) u_\varphi^i + (C^2 + D^2) u_r^i +$$

$$+ J^2 u_{rr}^i) + \text{first order terms of } u +$$

$$+ \frac{1}{J} (Ae_1^i + Be_2^i) \left( \frac{\hat{\mu} + \hat{\mu}'}{\rho} \frac{dp}{dt} + p_1 \rho \right)_\psi +$$

$$+ \frac{1}{J} (Ce_1^i + De_2^i) \left( \frac{\hat{\mu} + \hat{\mu}'}{\rho} \frac{dp}{dt} + p_1 \rho \right)_\varphi +$$

$$+ n^i \left( \frac{\hat{\mu} + \hat{\mu}'}{\rho} \frac{dp}{dt} + p_1 \rho \right)_r$$

$$= f^i - p_2 \theta_{x_1} + \frac{\hat{\mu} + \hat{\mu}'}{\rho} f_{x_1}^0 ,$$

where we note that  $J^2 = (AC + BD)^2 - (A^2 + B^2)(C^2 + D^2)$ . Now let  $\chi$  be any fixed function in  $C_0^\infty(O_J)$ . By methods similar to those of Lemma 5.5, we can get

**Lemma 5.6.** For any positive number  $\epsilon$  and  $1 \leq k \leq 4$ , it holds that

$$\begin{aligned} & \|\chi D_{\psi, \varphi}^k (\rho, u)(t)\|^2 + \int_0^t \|\chi D_{\psi, \rho}^k Du\|^2 + \|\chi D_{\psi, \rho}^k (\frac{dp}{dt})\|^2 d\tau \\ & \leq C \|\rho_0, u_0\|_k^2 + \int_0^t \epsilon \|\chi D_{\psi, \rho}^k \rho\|^2 + C(1 + \epsilon^{-1}) ( \\ & \quad \|u, \theta\|_k^2 + \|f^0\|_k^2 + \sum_{i=1}^3 \|f^i\|_{k-1}^2) d\tau + CN^3(0, t) . \end{aligned}$$

Moreover, by (5.8), we get:

$$\begin{aligned} (5.9) \quad & \frac{2\hat{\mu} + \hat{\mu}'}{\rho} \left( \frac{dp}{dt} \right)_r + p_1 \rho_r = -n^i u_t^i + \frac{\hat{\mu}}{J^2} ( \\ & (A^2 + B^2)n^i u_{\psi\psi}^i + 2(AC + BD)n^i u_{\psi\varphi}^i + (C^2 + D^2)n^i u_{\varphi\varphi}^i - \\ & - J(Ae_1^i + Be_2^i)u_{\psi r}^i - J(Ce_1^i + De_2^i)u_{r\varphi}^i ) + \end{aligned}$$

$$+ \text{first order terms of } (u, \theta) + n^{\frac{1}{2}} f^1 + \frac{2\hat{\mu} + \hat{\mu}^0}{\hat{\rho}} f_r^0 .$$

Estimating the equation

$$\int_0^t \int \chi^2 |D_{\psi, \varphi}^k D_x^l (\text{equation (5.9)})|^2 dx dt ,$$

for  $0 \leq k+l \leq 3$ , we have

**Lemma 5.7.** For  $0 \leq k+l \leq 3$ , it holds that

$$\begin{aligned} & \| \chi D_{\psi, \varphi}^k D_x^l \rho(t) \|^2 + \int_0^t \| \chi D_{\psi, \varphi}^k D_x^{l+1} \rho \|^2 + \\ & + \| \chi D_{\psi, \varphi}^k D_x^{l+1} \left( \frac{d\rho}{dt} \right) \|^2 dt \leq C \| \rho_0 \|_{k+l+1}^2 + \\ & + C \int_0^t \| \chi D_{\psi, \varphi}^{k+1} D_x^l Du \|^2 + \| u_t \|_{k+l}^2 + \| u, \theta \|_{k+l+1}^2 + \\ & + \| f^0 \|_{k+l+1}^2 + \sum_{i=1}^3 \| f^i \|_{k+l}^2 dt + CN^3(0, t) . \end{aligned}$$

Sixth, we note the following lemma about the linear stationary Stokes Equation (see [7], for example).

**Lemma 5.8.** Consider the problem

$$(5.10) \quad \begin{cases} \bar{\rho} u_j^j = g^0 , \\ -\mu u_j^1 + p_1 \rho_{x_1} = g^1 , \\ u|_{\partial\Omega} = 0 , \end{cases}$$

where  $g^0 \in H^{k+1}$  and  $g^1 \in H^k$  ( $k \geq 0$ ).

Then the problem (5.10) has a solution  $(\rho, u)$  such that  $\rho \in H^{k+1}$  and  $u \in H^{k+2} \cap H_0^1$  which is unique modulo a constant of integration for  $\rho$ ; this solution satisfies

$$\|u\|_{k+2}^2 + \|D\psi\|_k^2 \leq C(\|g^0\|_{k+1}^2 + \sum_{i=1}^3 \|g^i\|_k^2).$$

Apply Lemma 5.8 to the equations  $\{(3.4)\}_{i=0}^3$ , we have

Lemma 5.9. For  $2 \leq k \leq 5$ , it holds that

$$\|u\|_k^2 + \|D\psi\|_{k-2}^2 \leq C(\|\frac{dp}{dt}\|_{k-1}^2 + \|u_t\|_{k-2}^2 +$$

$$+ \|\theta\|_{k-1}^2 + \sum_{i=1}^3 \|f^i\|_{k-2}^2 + \|f^0\|_{k-1}^2).$$

Furthermore by differentiating the equations  $\{(3.4)\}_{i=0}^3$   $k$  times with respect to  $\psi$  and  $\varphi$  in each  $O_j$ , multiplying them by  $x$  and again applying Lemma 5.8, we obtain

Lemma 5.10. For  $1 \leq l+k \leq 3$ , it holds that

$$\begin{aligned} & \|x D^{2+l} D_{\psi, \varphi}^k u\|^2 + \|x D^{1+l} D_{\psi, \varphi}^k p\|^2 \\ & \leq C(\|x D^{1+l} D_{\psi, \varphi}^k (\frac{dp}{dt})\|^2 + \|u, \theta\|_{l+k+1}^2 + \|p\|_{l+k}^2 + \\ & \quad + \|p_t\|_{l+k}^2 + \|f^0\|_{k+l+1}^2 + \sum_{i=1}^3 \|f^i\|_{k+l}^2). \end{aligned}$$

Using all of the above preparations, we are ready to establish the desired a priori estimates. Although we omit the details, by combining Lemma 5.3 with  $k = 0$  and 1, Lemma 5.2 with  $k = 1, 2$  and 3, Lemma 5.4 with  $l = 0$  and 1, Lemma 5.5 with  $k = 1$  and 2, Lemma 5.6 with  $k = 1$  and 2, Lemma 5.7 with  $k = l = 0$  and  $k+l = 1$ , Lemma 5.9 with  $k = 2$  and 3 and Lemma 5.10 with  $l = 0$  and  $k = 1$ , and by choosing  $\epsilon$  suitably small, we have

$$(5.11) \quad \|p, u, \theta\|_2^2 + \|u_t, \theta_t\|_t^2 + \|p_t\|_1^2 +$$

$$\begin{aligned}
& + \int_0^t \|u_t, \theta\|_3^2 + \|\rho\|_2^2 + \|\frac{d\rho}{dt}\|_2^2 + \|u_{tt}, \theta_t\|_1^2 + \|\frac{d\rho}{dt} t\|_1^2 d\tau \\
& \leq C(\|\rho_0, u_0, \theta_0\|_2^2 + \sup_{0 \leq \tau \leq t} (\|f^0\|_1^2 + \sum_{i=1}^4 \|f^i\|_1^2) + \\
& + \int_0^t \|f^0\|_2^2 + \sum_{i=1}^4 \|f^i\|_1^2 + \|f_t^0\|_2^2 + \sum_{i=1}^4 \|f_{t-1}^i\|_1^2 d\tau) \\
& + CN^3(0, t) .
\end{aligned}$$

In the same way, after considering the situation differentiated with respect to  $t$ , we also have

$$\begin{aligned}
(5.12) \quad & \|\rho, u, \theta\|_t^2 + \|u_{tt}, \theta_{tt}\|_1^2 + \|\rho_{tt}\|_1^2 + \\
& + \int_0^t \|u_t, \theta_t\|_3^2 + \|\frac{d\rho}{dt}\|_2^2 + \|\frac{d\rho}{dt} t\|_2^2 + \|u_{tt}, \theta_{tt}\|_1^2 + \|\frac{d\rho}{dt} tt\|_2^2 d\tau \\
& \leq C(\|\rho_0, u_0, \theta_0\|_4^2 + \sum_{i=0}^4 \|f^i(0)\|_2^2 + \sup_{0 \leq \tau \leq t} (\sum_{i=1}^4 \|f_t^i\|_1^2 + \|f_t^0\|_1^2) + \\
& + \int_0^t \|f_t^0\|_2^2 + \sum_{i=1}^4 \|f_t^i\|_1^2 + \|f_{tt}^0\|_2^2 + \sum_{i=1}^4 \|f_{tt-1}^i\|_1^2 d\tau) + \\
& + CN^3(0, t) .
\end{aligned}$$

By beginning with the estimates (5.11) and (5.12) and combining the Lemmas 5.2 ~ 5.10 again, we can reach

$$\begin{aligned}
(5.13) \quad & \sup_{0 \leq \tau \leq t} (\|\rho, u, \theta\|_4^2 + \|u_t, \theta_t\|_2^2 + \|u_{tt}, \theta_{tt}\|_1^2 + \|\rho_t\|_3^2 + \\
& + \|\rho_{tt}\|_1^2) + \int_0^t \|u, \theta\|_5^2 + \|\rho\|_4^2 + \|u_t, \theta_t\|_3^2 + \|\frac{d\rho}{dt}\|_4^2 + \\
& + \|u_{tt}, \theta_{tt}\|_1^2 + \|\frac{d\rho}{dt} t\|_2^2 + \|\frac{d\rho}{dt} tt\|_2^2 d\tau \leq
\end{aligned}$$

$$\begin{aligned}
& \leq C(\|p_0, u_0, \theta_0\|_4^2 + \sup_{0 \leq \tau \leq t} (\|f^0\|_3^2 + \|f_t^0\|_1^2 + \sum_{i=1}^4 \|f_t^i\|_2^2 \\
& + \|f_t^1\|_2^2) + \int_0^t \|f^0\|_4^2 + \|f_t^0\|_2^2 + \|f_{tt}^0\|_2^2 + \sum_{i=1}^4 (\|f_t^i\|_3^2 + \|f_t^i\|_1^2 + \|f_{tt}^i\|_1^2) d\tau + N^3(0, t)) .
\end{aligned}$$

Therefore, noting that

$$\frac{dp}{dt} = p_t + \bar{\rho} u^j p_{x_j},$$

we have consequently

Lemma 5.11. It holds that

$$\begin{aligned}
(5.14) \quad N^2(0, t) & \leq C(\|p_0, u_0, \theta_0\|_4^2 + \sup_{0 \leq \tau \leq t} (\|f^0\|_3^2 + \|f_t^0\|_1^2 + \\
& + \sum_{i=1}^4 (\|f_t^i\|_2^2 + \|f_t^1\|_2^2)) + \int_0^t (\|f^0\|_4^2 + \|f_t^0\|_2^2 + \\
& \|f_{tt}^0\|_2^2 + \sum_{i=1}^4 (\|f_t^i\|_3^2 + \|f_t^i\|_1^2 + \|f_{tt}^i\|_1^2) d\tau + N^3(0, t)) .
\end{aligned}$$

Now let us establish the estimates for  $\{f^i\}$ .

Lemma 5.12. It holds that the right hand side of

$$(5.14) \leq C(\|p_0, u_0, \theta_0\|_4^2 + (N(0, t) + \|\theta\|_5)N^2(0, t)).$$

Here we note that Lemmas 5.11 and 5.12 easily imply the desired a priori estimates, in fact, we may choose  $\epsilon_1$  so small that

$$N^2(0, t) \leq C(\|p_0, u_0, \theta_0\|_4^2 \text{ for } N(0, t), \|\theta\|_5 \leq \epsilon_1).$$

Proof of Lemma 5.12. Because there are too many terms to estimate, we only give proofs for a few examples. The remaining terms can be proved in the same way.

Consider  $f^0$ , for example, as follows:

$$\begin{aligned}
& \sup_{\tau} \|(\bar{\rho} - \rho - \tilde{\rho}) u_{x_j}^j - \tilde{\rho}_{x_j} u_j^j\|_3^2 \\
& \leq \sup_{\tau} (\|\rho\|_3^2 \|u\|_4^2 + \|\tilde{\rho} - \bar{\rho}\|_3^2 \|u\|_4^2 + \|D\tilde{\rho}\|_3^2 \|u\|_3^2) \\
& \leq C E_0(N(0,t) + \|\Phi\|_5) N^2(0,t) ,
\end{aligned}$$

$$\begin{aligned}
& \int_0^t \|(\bar{\rho} - \rho - \tilde{\rho}) u_{x_j}^j - \tilde{\rho}_{x_j} u_j^j\|_4^2 d\tau \\
& \leq \int_0^t \|\rho\|_4^2 \|u\|_5^2 + \|\tilde{\rho} - \bar{\rho}\|_4^2 \|u\|_5^2 + \|D\tilde{\rho}\|_4^2 \|u\|_4^2 d\tau \\
& \leq C E_0(N(0,t) + \|\Phi\|_5) N^2(0,t) , \dots
\end{aligned}$$

and so on. Proceeding in this manner proves Lemma 5.12. Finally, in order to prove Theorem 1.1, it suffices to show the exponential decay of the solution. To do that, we may estimate the equations

$$\left\{
\begin{aligned}
L_u^0(e^{\alpha t}(\rho, u)) &= e^{\alpha t} f^0 + \alpha e^{\alpha t} \rho , \\
L^i(e^{\alpha t}(\rho, u, \theta)) &= e^{\alpha t} f^i + \alpha e^{\alpha t} u^i , \\
L^4(e^{\alpha t}(u, \theta)) &= e^{\alpha t} f^4 + \alpha e^{\alpha t} \theta ,
\end{aligned}
\right.$$

in the same way as (5.11) and taking  $\alpha$  suitably small.

REFERENCES

- [1] S. Agmon, A. Douglis, Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary condition I, Comm. Pure Appl. Math., 12, 1959, 623-727.
- [2] S. Agmon, A. Douglis, Nirenberg: Estimates near the boundar for solutions of elliptic partial differential equations satisfying boundary condition II, Comm. Appl. Math., 17, 1964, 35-92.
- [3] A. V. Vrazhikhov, V. V. Shelukkin: Unique global solution in time of initial-boundary value problems for one-dimensional equations of a viscous gas, Prikl. Mat. Mech., 41, 1977, 282-291.
- [4] A. Matsumura, T. Nishida: The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ., 20, 1980, 67-104.
- [5] A. Matsumura: An energy method for the equations of motion of compressible viscous and heat-conductive fluids, University of Wisconsin-Madison, MRC Technical Summary Report #2194.
- [6] A. Tani: On the first initial-boundary value problem of compressible viscous fluid motion, Publ. RIMS, Kyoto Univ., 13, 1977, 193-253.
- [7] R. Teman: Navier Stokes Equations, North-Holland Publishing Company-Amsterdam, New York, Oxford.

AM/TN/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2237 14 MR 17 1981 AD-A103866	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The Initial Boundary Value Problem for the Equations of Motion of Compressible Viscous and Heat-Conductive Fluid	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Akitaka Matsumura and Takaaki Nishida	8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 NSF-MCS-7927062	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis	
11. CONTROLLING OFFICE NAME AND ADDRESS  (see Item 18 below)	12. REPORT DATE July 1981	13. NUMBER OF PAGES 23
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED	
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709	National Science Foundation Washington, DC 20550	
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Initial boundary value problem, Equations of motion, Compressible viscous and heat-conductive fluid, Global solutions in time, Energy method.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We prove the global existence and uniqueness of solutions to the equations of motion for compressible, viscous and heat-conductive Newtonian fluid in a bounded domain, with small initial data and external force, and boundary conditions of zero velocity and constant temperature. We also show that the solution decays exponentially to a unique equilibrium state. The proof uses an energy method similar to the one used in our previous results on the pure initial value problem plus some new techniques for estimates near the boundary.		